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TOPICS IN SPECIAL FUNCTIONS

G. D. ANDERSON[1], M. K. VAMANAMURTHY, AND M. VUORINEN

Abstract. The authors survey recent results in special functions, particularly the gamma function and the Gaussian hypergeometric function.

1. Introduction

Conformal invariants are powerful tools in the study of quasiconformal mappings, and many of these have expressions in terms of special functions. For instance, the distortion results in geometric function theory, such as the quasiconformal Schwarz Lemma, involve special functions. A frequent task is to simplify complicated inequalities, so as to clarify the dependence on important parameters without sacrificing sharpness. For these reasons we were led to study, as an independent subject, various questions for special functions such as monotoneity properties and majorants/minorants in terms of rational functions. These new inequalities gave refined versions of some classical distortion theorems for quasiconformal maps. The classes of functions that occur include complete elliptic integrals, hypergeometric functions, and Euler's gamma function. The main part of our research is summarized in [AVV5].

In the later development most of our research has involved applications to geometric properties of quasiconformal maps. However, some of the questions concerning special functions, raised in [AVV1], [AVV3], and [AVV5], relate to special functions which are useful in geometric function theory in general, not just to quasiconformal maps. In this survey our goal is to review the latest developments of the latter type, due to many authors [A1]–[A9], [AlQ1, AlQ2, AW, BPR1, BPR2, BPS, BP, EL, K1, K2, Ku].

The methods used in these studies are based on classical analysis. One of the technical tools is the Monotone l'Hôpital's Rule, stated in the next paragraph, which played an important role in our work [AVV4]–[AVV5]. The authors discovered this result in [AVV4], unaware that it had been used earlier (without the name) as a technical tool in differential geometry. See [Ch, p. 124, Lemma 3.1] or [AQVV, p. 14] for relevant remarks.

1.1. **Lemma.** *For $-\infty < a < b < \infty$, let g and h be real-valued functions that are continuous on $[a,b]$ and differentiable on (a,b), with $h' \neq 0$ on (a,b).*

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If g'/h' is strictly increasing (resp. decreasing) on (a,b), then the functions

$$\frac{g(x)-g(a)}{h(x)-h(a)} \quad \textit{and} \quad \frac{g(x)-g(b)}{h(x)-h(b)}$$

are also strictly increasing (resp. decreasing) on (a,b).

Graphing of the functions and computer experiments in general played an important role in our work. For instance, the software that comes with the book [AVV5] provides computer programs for such experiments.

We begin this survey by discussing some recent results on the gamma function, including monotoneity and convexity properties and close approximations for the Euler-Mascheroni constant. Hypergeometric functions have a very central role in this survey. We give here a detailed proof of the so-called Elliott's identity for these functions following an outline suggested by Andrews, Askey, and Roy in [AAR, p. 138]. This identity contains, as a special case, the classical Legendre Relation and has been studied recently in [KV] and [BPSV]. After this we discuss mean values, a topic related to complete elliptic integrals and their estimation, and we present several sharp approximations for complete elliptic integrals. We display inequalities for hypergeometric functions that generalize the Landen relation, and conclude the paper with a remark on recent work of geometric mapping properties of hypergeometric functions as a function of a complex argument.

This survey does not cover recent work on the applications of special functions to the change of distance under quasiconformal maps. For this subject the interested reader may consult [AVV5].

2. The Γ and Ψ functions

Throughout this paper Γ will denote Euler's gamma function, defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \operatorname{Re} z > 0,$$

and then continued analytically to the finite complex plane minus the set of nonpositive integers. The recurrence formula $\Gamma(z+1) = z\,\Gamma(z)$ yields $\Gamma(n+1) = n!$ for any positive integer n. We also use the fact that $\Gamma(1/2) = \sqrt{\pi}$. The beta function is related to the gamma function by $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. The logarithmic derivative of the gamma function will be denoted, as usual, by

$$\Psi(z) \equiv \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

The Euler-Mascheroni constant γ is defined as (see [A2], [TY], [Y])

$$\gamma \equiv \lim_{n\to\infty} D_n = 0.5772156649\ldots;\ D_n \equiv \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

Then $\Psi(1) = \Gamma'(1) = -\gamma$ and $\Psi(1/2) = -\gamma - 2\log 2$. For a survey of the gamma function see [G], and for some inequalities for the gamma and psi functions see [A1].

2.1. **Approximation of the Euler-Mascheroni constant.** The convergence of the sequence D_n to γ is very slow (the speed of convergence is studied by Alzer [A2]). D. W. DeTemple [De] studied a modified sequence which converges faster and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad \text{where} \quad R_n \equiv \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right).$$

Now let

$$h(n) = R_n - \gamma, \quad H(n) = n^2 h(n), \; n \geqslant 1.$$

Since $\Psi(n) = -\gamma - 1/n + \sum_{k=1}^{n} 1/k$, we see that

$$H(n) = (R_n - \gamma)n^2 = (\Psi(n) + 1/n - \log(n + \frac{1}{2}))n^2.$$

Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1, \infty)$ from $H(1) = -\gamma + 1 - \log(3/2) = 0.0173\ldots$ to $1/24 = 0.0416\ldots$. E. A. Karatsuba proved in [K1] that for all integers $n \geqslant 1, H(n) < H(n+1)$, by clever use of Stirling's formula and Fourier series. Moreover, using the relation $\gamma = 1 - \Gamma'(2)$ she obtained, for $k \geq 1$,

$$-c_k \leq \gamma - 1 + (\log k) \sum_{r=1}^{12k+1} d(k,r) - \sum_{r=1}^{12k+1} \frac{d(k,r)}{r+1} \leq c_k,$$

where

$$c_k = \frac{2}{(12k)!} + 2k^2 e^{-k}, \quad d(k,r) = (-1)^{r-1} \frac{k^{r+1}}{(r-1)!(r+1)},$$

giving exponential convergence. Some computer experiments also seem to indicate that $((n+1)/n)^2 H(n)$ is a decreasing convex function.

2.2. **Gamma function and volumes of balls.** Formulas for geometric objects, such as volumes of solids and arc lengths of curves, often involve special functions. For example, if Ω_n denotes the volume of the unit ball $B^n = \{x : |x| < 1\}$ in $\mathbb{R}^n$, and if ω_{n-1} denotes the $(n-1)$-dimensional surface area of the unit sphere $S^{n-1} = \{x : |x| = 1\}$, $n \geqslant 2$, then

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma((n/2)+1)}\,; \; \omega_{n-1} = n\Omega_n.$$

It is well known that for $n \geqslant 7$ both Ω_n and ω_n decrease to 0 (cf. [AVV5, 2.28]). However, neither Ω_n nor ω_n is monotone for n on $[2, \infty)$. On the other hand, $\Omega_n^{1/(n \log n)}$ decreases to $e^{-1/2}$ as $n \to \infty$ [AVV1, Lemma 2.40(2)].

Recently H. Alzer [A4] has obtained the best possible constants a, b, A, B, α, β such that

$$a\,\Omega_{n+1}^{\frac{n}{n+1}} \leqslant \Omega_n \leqslant b\,\Omega_{n+1}^{\frac{n}{n+1}},$$
$$\sqrt{\frac{n+A}{2\pi}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+B}{2\pi}},$$
$$\left(1 + \frac{1}{n}\right)^{\alpha} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1 + \frac{1}{n}\right)^{\beta}$$

for all integers $n \geqslant 1$. He showed that $a = 2/\sqrt{\pi} = 1.12837\ldots$, $b = \sqrt{e} = 1.64872\ldots$, $A = 1/2$, $B = \pi/2 - 1 = 0.57079\ldots$, $\alpha = 2 - (\log \pi)/\log 2 = 0.34850\ldots$, $\beta = 1/2$. For some related results, see [KlR].

2.3. **Monotoneity properties.** In [AnQ] it is proved that the function

$$f(x) \equiv \frac{\log \Gamma(x+1)}{x \log x} \tag{2.4}$$

is strictly increasing from $(1,\infty)$ onto $(1-\gamma, 1)$. In particular, for $x \in (1,\infty)$,

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}. \tag{2.5}$$

The proof required the following two technical lemmas, among others:

2.6. **Lemma.** *The function*

$$g(x) \equiv \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3}$$

is positive for $x \in [1, 4)$.

2.7. **Lemma.** *The function*

$$h(x) \equiv x^2\, \Psi'(1+x) - x\, \Psi(1+x) + \log \Gamma(1+x) \tag{2.8}$$

is positive for all $x \in [1,\infty)$.

It was conjectured in [AnQ] that the function f in (2.4) is concave on $(1,\infty)$.

2.9. Horst Alzer [A2] has given an elegant proof of the monotoneity of the function f in (2.4) by using the Monotone l'Hôpital's Rule and the convolution theorem for Laplace transforms. In a later paper [A3] he has improved the estimates in (2.5) to

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma}, \quad x \in (0,1), \tag{2.10}$$

where $\alpha \equiv 1 - \gamma = 0.42278\ldots$, $\beta \equiv \frac{1}{2}\left(\pi^2/6 - \gamma\right) = 0.53385\ldots$ are best possible. If $x \in (1,\infty)$, he also showed that (2.10) holds with best constants $\alpha \equiv \frac{1}{2}\left(\pi^2/6 - \gamma\right) = 0.53385\ldots$, $\beta \equiv 1$.

2.11. Elbert and Laforgia [EL] have shown that the function g in Lemma 2.6 is positive for all $x > -1$. They used this result to prove that the function h in Lemma 2.7 is strictly decreasing from $(-1, 0]$ onto $[0,\infty)$ and strictly increasing from $[0,\infty)$ onto $[0,\infty)$. They also showed that $f'' < 0$ for $x > 1$, thus proving the Anderson-Qiu conjecture [AnQ], where f is as in (2.4).

2.12. Berg and Pedersen [BP] have shown that the function f in (2.4) is not only strictly increasing from $(0,\infty)$ onto $(0, 1)$, but is even a (nonconstant) so-called *Bernstein function*. That is, $f > 0$ and f' is completely monotonic, i.e., $f' > 0$, $f'' < 0$, $f''' > 0$, $\ldots$. In particular, the function f is strictly increasing and strictly concave on $(0,\infty)$.

In fact, they have proved the stronger result that $1/f$ is a Stieltjes transform, that is, can be written in the form

$$\frac{1}{f(x)} = c + \int_0^\infty \frac{d\sigma(t)}{x+t}, \quad x > 0,$$

where the constant c is non-negative and σ is a non-negative measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{d\sigma(t)}{1+t} < \infty.$$

In particular, for $1/f$ they have shown by using Stirling's formula that $c = 1$. Also they have obtained $d\sigma(t) = H(t)dt$, where H is the continuous density

$$H(t) = \begin{cases} t\dfrac{\log|\Gamma(1-t)| + (k-1)\log t}{(\log|\Gamma(1-t)|)^2 + (k-1)^2\pi^2}, & t \in (k-1,k), k = 1,2,\dots, \\ 0 & , \quad t = 1,2,\dots. \end{cases}$$

Here log denotes the usual natural logarithm. The density $H(t)$ tends to $1/\gamma$ as t tends to 0, and σ has no mass at 0.

2.13**.** In "The Lost Notebook and Other Unpublished Papers" of Ramanujan [Ra1], the Indian mathematical genius, appears the following record:

$$\text{“}\Gamma(1+x) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left\{8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right\}^{1/6},$$

where θ_x is a positive proper fraction

$$\begin{array}{ll} \theta_0 = \dfrac{30}{\pi^3} = .9675 & \\ \theta_{1/12} = .8071 & \theta_{7/12} = .3058 \\ \theta_{2/12} = .6160 & \theta_{8/12} = .3014 \\ \theta_{3/12} = .4867 & \theta_{9/12} = .3041 \\ \theta_{4/12} = .4029 & \theta_{10/12} = .3118 \\ \theta_{5/12} = .3509 & \theta_{11/12} = .3227 \\ \theta_{6/12} = .3207 & \theta_1 = .3359 \\ \theta_\infty = 1.\text{”} & \end{array}$$

Of course, the values in the above table, except θ_∞, are irrational and hence the decimals should be nonterminating as well as nonrecurring. The record stated above has been the subject of intense investigations and is reviewed in [BCK], page 48 (Question 754). This note of Ramanujan led the authors of [AVV5] to make the following conjecture.

2.14. **Conjecture.** Let

$$G(x) = (e/x)^x \Gamma(1+x)/\sqrt{\pi}$$

and

$$H(x) = G(x)^6 - 8x^3 - 4x^2 - x = \frac{\theta_x}{30}.$$

Then H is increasing from $(1, \infty)$ into $(1/100, 1/30)$ [AVV5, p. 476].

2.15**.** In a nice piece of work, E. A. Karatsuba [K2] has proved the above conjecture. She did this by representing the function $H(x)$ as an integral for which she was able to find an asymptotic development. Her work also led to

an interesting asymptotic formula for the gamma function:

$$\Gamma(x+1) = \sqrt{\pi}\Big(\frac{x}{e}\Big)^x\Big(8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}$$
$$-\frac{9511}{403200x^4}-\frac{10051}{716800x^5}+\frac{47474887}{1277337600x^6}+\frac{a_7}{x^7}+\cdots+\frac{a_n}{x^n}+\Delta_{n+1}(x)\Big)^{1/6},$$

where $\Delta_{n+1}(x)=O(\frac{1}{x^{n+1}})$, as $x\to\infty$, and where each a_k is given explicitly in terms of the Bernoulli numbers.

3. Hypergeometric functions

Given complex numbers a, b, and c with $c\neq 0,-1,-2,\ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C}\setminus[1,\infty)$ of

$$F(a,b;c;z)={}_2F_1(a,b;c;z)\equiv\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{z^n}{n!},\quad |z|<1. \tag{3.1}$$

Here $(a,0)=1$ for $a\neq 0$, and (a,n) is the *shifted factorial function*

$$(a,n)\equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n=1,2,3,\ldots$.

The hypergeometric function $w=F(a,b;c;z)$ in (3.1) has the simple differentiation formula

$$\frac{d}{dz}F(a,b;c;z)=\frac{ab}{c}F(a+1,b+1;c+1;z). \tag{3.2}$$

The behavior of the hypergeometric function near $z=1$ in the three cases $a+b<c$, $a+b=c$, and $a+b>c$, $a,b,c>0$, is given by

$$\begin{cases} F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ a+b<c,\\ B(a,b)F(a,b;a+b;z)+\log(1-z)\\ \qquad\qquad =R(a,b)+O((1-z)\log(1-z)),\\ F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z),\ c<a+b, \end{cases} \tag{3.3}$$

where $R(a,b)=-2\gamma-\Psi(a)-\Psi(b)$, $R(a)\equiv R(a,1-a)$, $R(\frac{1}{2})=\log 16$, and where log denotes the principal branch of the complex logarithm. The above asymptotic formula for the *zero-balanced* case $a+b=c$ is due to Ramanujan (see [As], [Be1]). This formula is implied by [AS, 15.3.10].

The asymptotic formula (3.3) gives a precise description of the behavior of the function $F(a,b;a+b;z)$ near the logarithmic singularity $z=1$. This singularity can be removed by an exponential change of variables and the transformed function will be nearly linear.

3.4. **Theorem.** [AQVV] *For $a,b>0$, let $k(x)=F(a,b;a+b;1-e^{-x})$, $x>0$. Then k is an increasing and convex function with $k'((0,\infty))=(ab/(a+b),\ \Gamma(a+b)/(\Gamma(a)\Gamma(b)))$.*

3.5. **Theorem.** [AQVV] *Given $a,b>0$, and $a+b>c$, $d\equiv a+b-c$, the function $\ell(x)=F(a,b;c;1-(1+x)^{-1/d})$, $x>0$, is increasing and convex, with $\ell'((0,\infty))=(ab/(cd),\ \Gamma(c)\Gamma(d)/(\Gamma(a)\Gamma(b)))$.*

3.6. **Gauss contiguous relations and derivative formula.** The six functions $F(a\pm 1,b;c;z)$, $F(a,b\pm 1;c;z)$, $F(a,b;c\pm 1;z)$ are called *contiguous* to $F(a,b;c;z)$. Gauss discovered 15 relations between $F(a,b;c;z)$ and pairs of its contiguous functions [AS, 15.2.10–15.2.27], [R2, Section 33]. If we apply these relations to the differentiation formula (3.2), we obtain the following useful formulas.

3.7. **Theorem.** *For $a,b,c>0$, $z\in(0,1)$, let $u=u(z)=F(a-1,b;c;z)$, $v=v(z)=F(a,b;c;z)$, $u_1=u(1-z)$, $v_1=v(1-z)$. Then*

$$z\frac{du}{dz}=(a-1)(v-u), \tag{3.8}$$

$$z(1-z)\frac{dv}{dz}=(c-a)u+(a-c+bz)v, \tag{3.9}$$

and

$$\frac{ab}{c}z(1-z)F(a+1,b+1;c+1;z)=(c-a)u+(a-c+bz)v. \tag{3.10}$$

Furthermore,

$$z(1-z)\frac{d}{dz}\big(uv_1+u_1v-vv_1\big)=(1-a-b)\big[(1-z)uv_1-zu_1v-(1-2z)vv_1)\big]. \tag{3.11}$$

Formulas (3.8)-(3.10) in Theorem 3.7 are well known. See, for example, [AAR, 2.5.8]. On the other hand, formula (3.11), which follows from (3.8)-(3.9) is first proved in [AQVV, 3.13 (4)].

Note that the formula

$$z(1-z)\frac{dF}{dz}=(c-b)F(a,b-1;c;z)+(b-c+az)F(a,b;c;z) \tag{3.12}$$

follows from (3.9) if we use the symmetry property $F(a,b;c;z)=F(b,a;c;z)$.

3.13. **Corollary.** *With the notation of Theorem 3.7, if $a\in(0,1)$, $b=1-a<c$, then*

$$uv_1+u_1v-vv_1=u(1)=\frac{(\Gamma(c))^2}{\Gamma(c+a-1)\Gamma(c-a+1)}.$$

4. Hypergeometric differential equation

The function $F(a,b;c;z)$ satisfies the hypergeometric differential equation

$$z(1-z)w''+[c-(a+b+1)z]w'-abw=0. \tag{4.1}$$

Kummer discovered solutions of (4.1) in various domains, obtaining 24 in all; for a complete list of his solutions see [R2, pp. 174, 175].

4.2. **Lemma.** (1) *If $2c=a+b+1$ then both $F(a,b;c;z)$ and $F(a,b;c;1-z)$ satisfy* (4.1) *in the lens-shaped region $\{z: 0<|z|<1,\ 0<|1-z|<1\}$.* (2) *If $2c=a+b+1$ then both $F(a,b;c;z^2)$ and $F(a,b;c;1-z^2)$ satisfy the differential equation*

$$z(1-z^2)w''+[2c-1-(2a+2b+1)z^2]w'-4abzw=0 \tag{4.3}$$

in the common part of the disk $\{z:|z|<1\}$ and the lemniscate $\{z:|1-z^2|<1\}$.

Proof. By Kummer (cf. [R2, pp. 174-177]), the functions $F(a,b;c;z)$ and $F(a,b;a+b+1-c;1-z)$ are solutions of (4.1) in $\{z : 0 < |z| < 1\}$ and $\{z : 0 < |1-z| < 1\}$, respectively. But $a+b+1-c = c$ under the stated hypotheses. The result (2) follows from result (1) by the chain rule. □

4.4. **Lemma.** *The function $F(a,b;c;\sqrt{1-z^2})$ satisfies the differential equation*

$$Z^3(1-Z)zw'' - \{Z(1-Z) + [c-(a+b+1)Z]Zz^2\}w' - abz^3w = 0,$$

in the subregion of the right half-plane bounded by the lemniscate $r^2 = 2\cos(2\vartheta)$, $-\pi/4 \le \vartheta \le \pi/4$, $z = re^{i\vartheta}$. Here $Z = \sqrt{1-z^2}$, where the square root indicates the principal branch.

Proof. From (4.1), the differential equation for $w = F(a,b;c;t)$ is given by

$$t(1-t)\frac{d^2w}{dt^2} + [c-(a+b+1)t]\frac{dw}{dt} - abw = 0.$$

Now put $t = \sqrt{1-z^2}$. Then

$$\frac{dz}{dt} = -\frac{t}{z},\ \frac{dt}{dz} = -\frac{z}{t},\ \frac{d^2t}{dz^2} = -\frac{1}{t^3}$$

and

$$\frac{dw}{dt} = -\frac{t}{z}\frac{dw}{dz},\ \frac{d^2w}{dt^2} = \frac{t^2}{z^2}\frac{d^2w}{dz^2} - \frac{1}{z^3}\frac{dw}{dz}.$$

So

$$t(1-t)\Big[\frac{t^2}{z^2}w'' - \frac{1}{z^3}w'\Big] + \Big[c-(a+b+1)t\Big]\Big(-\frac{t}{z}\Big)w' - abw = 0.$$

Multiplying through by z^3 and replacing t by $Z \equiv \sqrt{1-z^2}$ gives the result. □

If w_1 and w_2 are two solutions of a second order differential equation, then their *Wronskian* is defined to be $W(w_1,w_2) \equiv w_1w_2' - w_2w_1'$.

4.5. **Lemma.** [AAR, Lemma 3.2.6] *If w_1 and w_2 are two linearly independent solutions of* (4.1)*, then*

$$W(z) = W(w_1,w_2)(z) = \frac{A}{z^c(1-z)^{a+b-c+1}},$$

where A is a constant.

(Note the misprint in [AAR, (3.10)], where the coefficient $x(1-x)$ is missing from the first term.)

4.6. **Lemma.** *If $2c = a+b+1$ then, in the notation of Theorem* 3.7*,*

$$(c-a)(uv_1 + u_1v) + (a-1)vv_1 = A\cdot z^{1-c}(1-z)^{1-c}. \tag{4.7}$$

Proof. If $2c = a+b+1$ then by Lemma 4.2(1), both $v(z)$ and $v(1-z)$ are solutions of (4.1). Since $W(z) = W(v_1,v)(z) = v'(z)v_1(z) - v(z)v_1'(z)$, we have

$$\begin{aligned} z(1-z)W(z) &= z(1-z)(v'v_1 - vv_1') \\ &= (c-a)(uv_1 + u_1v) + (2a+b-2c)vv_1 \\ &= (c-a)(uv_1 + u_1v) + (a-1)vv_1. \end{aligned}$$

Next, since $2c = a+b+1$, Lemma 4.5 shows that $z^c(1-z)^c W(z) = A$, and the result follows. $\square$

Note that in the particular case $c = 1, a = b = \frac{1}{2}$ the right side of (4.7) is constant and the result is similar to Corollary 3.13. This particular case is Legendre's Relation (5.3), and this proof of it is due to Duren [Du].

4.8. **Lemma.** *If $a, b > 0, c \geq 1$, and $2c = a+b+1$, then the constant A in Lemma 4.6 is given by $A = (\Gamma(c))^2/(\Gamma(a)\Gamma(b))$. In particular, if $c = 1$ then Lemma 4.6 reduces to Legendre's Relation (5.8) for generalized elliptic integrals.*

Proof. The idea of the proof is to replace the possibly unbounded hypergeometric functions in formula (4.7) by bounded or simpler ones. Therefore we consider three cases.

*C*ase (1): $c \geq 2$. Now $a+b \geq c+1 \geq 3$. By (3.3) or [AS, 15.3.3], we have

$$u(z) = (1-z)^{2-c}F(c+1-a, c-b; c; z), \quad u_1(z) = z^{2-c}F(c+1-a, c-b; c; 1-z),$$

$$v(z) = (1-z)^{1-c}F(c-a, c-b; c; z), \quad v_1(z) = z^{1-c}F(c-a, c-b; c; 1-z).$$

Hence

$$\begin{aligned} A = (c-a)[(1-z)F(c+1-a, c-b; c; z)F(c-a, c-b; c; 1-z) \\ +zF(c+1-a, c-b; c; 1-z)F(c-a, c-b; c; z)] \\ +(a-1)F(c-a, c-b; c; z)F(c-a, c-b; c; 1-z). \end{aligned}$$

Now, since $a+b-c = c-1$, letting $z \to 0$, from (3.3) we get

$$\begin{aligned} A &= (c-a)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b))} + (a-1)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} \\ &= (c-1)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} = \frac{(\Gamma(c))^2}{\Gamma(a)\Gamma(b)}, \end{aligned}$$

as claimed.

*C*ase (2): $1 < c < 2$. Now, $1 < c < a+b < c+1 < 3$. Then

$$\begin{aligned} A = (c-a)[(1-z)^{c-1}u(z)F(c-a, c-b; c; 1-z) + z^{c-1}u_1(z)F(c-a, c-b; c; z)] \\ +(a-1)F(c-a, c-b; c; z)F(c-a, c-b; c; 1-z). \end{aligned}$$

Now letting $z \to 0$, from (3.3), as in Case (1), we get

$$\begin{aligned} A &= (c-a)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} + (a-1)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} \\ &= (c-1)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} = \frac{(\Gamma(c))^2}{\Gamma(a)\Gamma(b)}, \end{aligned}$$

as claimed.

*C*ase (3): $c = 1$. Now $a+b = 1$. Then

$$\begin{aligned} A &= (1-a)[u(z)v_1(z) + u_1(z)v(z) - v(z)v_1(z)] \\ &= (1-a)u_1(z)v(z) + (1-a)v_1(z)[u(z) - v(z)]. \end{aligned}$$

From [R1, Ex. 21(4), p.71] we have

$$\begin{aligned} u(z)-v(z) &= F(a-1,b;c;z)-F(a,b;c;z) \\ &= \frac{c-b}{c}zF(a,b;c+1;z)-zF(a,b;c;z), \end{aligned}$$

so that

$$\frac{u(z)-v(z)}{z}=\frac{c-b}{c}F(a,b;c+1;z)-F(a,b;c;z)\to -b/c,$$

as $z\to 0$. Also, by (3.3), $zv_1(z)\to 0$ as $z\to 0$. Hence, letting $z\to 0$, we get

$$\begin{aligned} A &= (1-a)u_1(1)=(1-a)\frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c+1-a)\Gamma(c-b)} \\ &= (1-a)\frac{(\Gamma(c))^2}{(1-a)\Gamma(a)\Gamma(b)}=\frac{(\Gamma(c))^2}{\Gamma(a)\Gamma(b)}, \end{aligned}$$

as claimed.

Note that, in Case (3), $\Gamma(c)=\Gamma(1)=1, \Gamma(b)=\Gamma(1-a)$, and thus by [AS, 6.1.17] $A=1/(\Gamma(a)\Gamma(1-a))=(\sin\pi a)/\pi$. □

For rational triples (a,b,c) there are numerous cases where the hypergeometric function $F(a,b;c;z)$ reduces to a simpler function (see [PBM]). Other important particular cases are *generalized elliptic integrals*, which we will now discuss. For $a,r\in(0,1)$, the *generalized elliptic integral of the first kind* is given by

$$\begin{aligned} \mathcal{K}_a=\mathcal{K}_a(r) &= \frac{\pi}{2}F(a,1-a;1;r^2) \\ &= (\sin\pi a)\int_0^{\pi/2}(\tan t)^{1-2a}(1-r^2\sin^2 t)^{-a}\,dt, \\ \mathcal{K}'_a=\mathcal{K}'_a(r) &= \mathcal{K}_a(r'). \end{aligned}$$

We also define

$$\mu_a(r)=\frac{\pi}{2\sin(\pi a)}\frac{\mathcal{K}'_a(r)}{\mathcal{K}_a(r)},\quad r'=\sqrt{1-r^2}.$$

The *invariant* of the linear differential equation

$$w''+pw'+qw=0, \tag{4.9}$$

where p and q are functions of z, is defined to be

$$I\equiv q-\frac{1}{2}p'-\frac{1}{4}p^2$$

(cf. [R2,p.9]). If w_1 and w_2 are two linearly independent solutions of (4.9), then their quotient $w\equiv w_2/w_1$ satisfies the differential equation

$$S_w(z)=2I,$$

where S_w is the Schwarzian derivative

$$S_w\equiv\left(\frac{w''}{w'}\right)'-\frac{1}{2}\left(\frac{w''}{w'}\right)^2$$

and the primes indicate differentiations (cf. [R2, pp. 18,19]).

From these considerations and the fact that $\mathcal{K}_a(r)$ and $\mathcal{K}'_a(r)$ are linearly independent solutions of (4.3) (see [AQVV, (1.11)]), it follows that $w = \mu_a(r)$ satisfies the differential equation

$$S_w(r) = \frac{-8a(1-a)}{(r')^2} + \frac{1+6r^2-3r^4}{2r^2(r')^4}.$$

The *generalized elliptic integral of the second kind* is given by

$$\begin{aligned}
\mathcal{E}_a = \mathcal{E}_a(r) &\equiv \frac{\pi}{2}F(a-1,1-a;1;r^2)\\
&= (\sin \pi a)\int_0^{\pi/2}(\tan t)^{1-2a}(1-r^2\sin^2 t)^{1-a}\,dt\\
\mathcal{E}'_a = \mathcal{E}'_a(r) &= \mathcal{E}_a(r'),\\
\mathcal{E}_a(0) = \frac{\pi}{2}, &\quad \mathcal{E}_a(1) = \frac{\sin(\pi a)}{2(1-a)}.
\end{aligned}$$

For $a = \frac{1}{2}$, $\mathcal{K}_a$ and $\mathcal{E}_a$ reduce to $\mathcal{K}$ and $\mathcal{E}$, respectively, the usual elliptic integrals of the first and second kind, respectively. Likewise $\mu_{1/2}(r) = \mu(r)$, the modulus of the well-known Grötzsch ring in the plane [LV].

4.10. **Corollary.** *The generalized elliptic integrals $\mathcal{K}_a$ and $\mathcal{E}_a$ satisfy the differential equations*

$$r(r')^2\frac{d^2\mathcal{K}_a}{dr^2} + (1-3r^2)\frac{d\mathcal{K}_a}{dr} - 4a(1-a)r\mathcal{K}_a = 0, \tag{4.11}$$

$$r(r')^2\frac{d^2\mathcal{E}_a}{dr^2} + (r')^2\frac{d\mathcal{E}_a}{dr} + 4(1-a)^2 r\mathcal{E}_a = 0, \tag{4.12}$$

respectively.

Proof. These follow from (4.3). $\square$

For $a = \frac{1}{2}$ these reduce to well-known differential equations [AVV5, pp. 474-475], [BF].

5. Identities of Legendre and Elliott

In geometric function theory the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ play an important role. These integrals may be defined, respectively, as

$$\mathcal{K}(r) = \tfrac{\pi}{2}F(\tfrac{1}{2},\tfrac{1}{2};1;r^2),\ \mathcal{E}(r) = \tfrac{\pi}{2}F(\tfrac{1}{2},-\tfrac{1}{2};1;r^2),$$

for $-1 < r < 1$. These are $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$, respectively, with $a = \frac{1}{2}$. We also consider the functions

$$\begin{aligned}
\mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), &\quad 0 < r < 1,\\
\mathcal{K}(0) = \pi/2, &\quad \mathcal{K}(1^-) = +\infty,
\end{aligned}$$

and

$$\mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \quad 0 \leqslant r \leqslant 1,$$

where $r' = \sqrt{1-r^2}$. For example, these functions occur in the following quasiconformal counterpart of the Schwarz Lemma [LV]:

5.1. **Theorem.** *For $K \in [1,\infty)$, let w be a K-quasiconformal mapping of the unit disk $D = \{z : |z| < 1\}$ into the unit disk $D' = \{w : |w| < 1\}$ with $w(0) = 0$. Then*

$$|w(z)| \leqslant \varphi_K(|z|),$$

where

$$\varphi_K(r) \equiv \mu^{-1}\big(\frac{1}{K}\mu(r)\big) \quad \textit{and} \quad \mu(r) \equiv \frac{\pi\mathcal{K}'(r)}{2\mathcal{K}(r)}. \tag{5.2}$$

This result is sharp in the sense that for each $z \in D$ and $K \in [1,\infty)$ there is an extremal K-quasiconformal mapping that takes the unit disk D onto the unit disk D' with $w(0) = 0$ and $|w(z)| = \varphi_K(|z|)$ (see [LV, p. 63]*).*

It is well known [BF] that the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ satisfy the Legendre relation

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}. \tag{5.3}$$

For several proofs of (5.3) see [Du].

In 1904, E. B. Elliott [E] (cf. [AVV3]) obtained the following generalization of this result.

5.4. **Theorem.** *If $a, b, c \geqslant 0$ and $0 < x < 1$ then*

$$F_1F_2 + F_3F_4 - F_2F_3 = \frac{\Gamma(a+b+1)\Gamma(b+c+1)}{\Gamma(a+b+c+\frac{3}{2})\Gamma(b+\frac{1}{2})}. \tag{5.5}$$

where

$$F_1 = F\left(\frac{1}{2}+a, -\frac{1}{2}-c; 1+a+b; x\right),$$
$$F_2 = F\left(\frac{1}{2}-a, \frac{1}{2}+c; 1+b+c; 1-x\right),$$
$$F_3 = F\left(\frac{1}{2}+a, \frac{1}{2}-c; 1+a+b; x\right),$$
$$F_4 = F\left(-\frac{1}{2}-a, \frac{1}{2}+c; 1+b+c; 1-x\right).$$

Clearly (5.3) is a special case of (5.5), when $a = b = c = 0$ and $x = r^2$. For a discussion of generalizations of Legendre's Relation see Karatsuba and Vuorinen [KV] and Balasubramanian, Ponnusamy, Sunanda Naik, and Vuorinen [BPSV].

Elliott proved (5.5) by a clever change of variables in multiple integrals. Another proof was suggested without details in [AAR, p. 138], and here we provide the missing details.

Proof of Theorem 5.4. In particular, let $y_1 \equiv F_3$, $y_2 \equiv x^{-a-b}(1-x)^{b+c}F_2$. Then by [R2, pp. 174, 175] or [AAR, (3.2.12), (3.2.13)], y_1 and y_2 are linearly independent solutions of (4.1).

By (3.12),

$$x(1-x)y_1' = \big(a+b+c+\frac{1}{2}\big)F_1 + [-(a+b+c+\frac{1}{2}) + (a+\frac{1}{2})x]F_3, \tag{5.6}$$

and by (3.9),

$$\begin{aligned} x(1-x)y_2' =& x(1-x)\big[-(a+b)x^{-a-b-1}(1-x)^{b+c} \\ & -(b+c)x^{-a-b}(1-x)^{b+c-1}\big]F_2 \\ & -x^{-a-b}(1-x)^{b+c}\big[\big(a+b+c+\frac{1}{2}\big)F_4 \\ & +\big[-(a+b+c+\frac{1}{2}\big)+\big(c+\frac{1}{2}\big)(1-x))F_2\big]. \end{aligned} \tag{5.7}$$

Multiplying (5.7) by y_1 and (5.6) by y_2 and subtracting, we obtain

$$\begin{aligned} x(1-x)(y_2y_1'-y_1y_2') &= \big(a+b+c+\frac{1}{2}\big)x^{-a-b}(1-x)^{b+c}F_1F_2 \\ &+\big[-\big(a+b+c+\frac{1}{2}\big)+\big(a+\frac{1}{2}\big)x\big]x^{-a-b}(1-x)^{b+c}F_2F_3 \\ &+x(1-x)[(a+b)x^{-a-b-1}(1-x)^{b+c} \\ &+(b+c)x^{-a-b}(1-x)^{b+c-1}]F_2F_3 \\ &+x^{-a-b}(1-x)^{b+c}\big(a+b+c+\frac{1}{2}\big)F_3F_4 \\ &+x^{-a-b}(1-x)^{b+c}\big[-(a+b+c+\frac{1}{2}\big)+\big(c+\frac{1}{2}\big)(1-x)\big]F_2F_3 \\ &= \big(a+b+c+\frac{1}{2}\big)x^{-a-b}(1-x)^{b+c}F_1F_2 \\ &+x^{-a-b}(1-x)^{b+c}\big[-(a+b+c+\frac{1}{2}\big) \\ &+\big(a+\frac{1}{2}\big)x+(a+b)(1-x) \\ &+(b+c)x-\big(a+b+c+\frac{1}{2}\big)+\big(c+\frac{1}{2}\big)(1-x)\big]F_2F_3 \\ &+x^{-a-b}(1-x)^{b+c}\big(a+b+c+\frac{1}{2}\big)F_3F_4 \\ &= \big(a+b+c+\frac{1}{2}\big)x^{-a-b}(1-x)^{b+c}F_1F_2 \\ &+x^{-a-b}(1-x)^{b+c}\big[-\big(a+b+c+\tfrac{1}{2}\big)\big]F_2F_3 \\ &+\big(a+b+c+\frac{1}{2}\big)x^{-a-b}(1-x)^{b+c}F_3F_4. \end{aligned}$$

So

$$\begin{aligned} x(1-x)W(y_2,y_1) &= \frac{A}{x^{a+b}(1-x)^{-b-c}} \\ &= x^{-a-b}(1-x)^{b+c}\big(a+b+c+\frac{1}{2}\big)[F_1F_2+F_3F_4-F_2F_3] \end{aligned}$$

by Lemma 4.5. Thus

$$F_1F_2+F_3F_4-F_2F_3=A,$$

where A is a constant.

Now, by (3.3),

$$F_1F_2 \text{ tends to } F\Big(\frac{1}{2}+a, -\frac{1}{2}-c; a+b+1; 1\Big) = \frac{\Gamma(a+b+1)\Gamma(b+c+1)}{\Gamma(b+\frac{1}{2})\Gamma(a+b+c+\frac{3}{2})}$$

as $x \to 1$, since $\frac{1}{2}+a+(-\frac{1}{2}-c) = a-c < a+b+1$.

Next

$$F_3F_4 - F_3F_2 = F_3(F_4 - F_2),$$

where $F_4 - F_2 \sim$ const $\cdot\ (1-x)^2 + O((1-x)^3)$, and

$$F_3 = \frac{\Gamma(a+b+1)\Gamma(b+c+1)}{\Gamma(b+\frac{1}{2})\Gamma(a+b+c+\frac{1}{2})}$$

if $a+\frac{1}{2}+\frac{1}{2}-c < a+b+1$, or $-c < b$, i.e., $b > 0$ or $c > 0$. If $-c = b = 0$, then

$$F_3 = \frac{R(a+\frac{1}{2}, \frac{1}{2}-c)}{B(a+\frac{1}{2}, \frac{1}{2}-c)} + O((1-x)\log(1-x))$$

by (3.3). In either case the product $F_3(F_4-F_2)$ tends to 0 as $x \to 1$. The third case $a+\frac{1}{2}+\frac{1}{2}-c > a+b+1$ is impossible since we are assuming that b, c are nonnegative. Thus $A = \Gamma(a+b+1)\Gamma(b+c+1)/(\Gamma(b+\frac{1}{2})\Gamma(a+b+c+\frac{3}{2}))$, as desired. □

The generalized elliptic integrals satisfy the identity

$$\mathcal{E}_a\mathcal{K}_a' + \mathcal{E}_a'\mathcal{K}_a - \mathcal{K}_a\mathcal{K}_a' = \frac{\pi\sin(\pi a)}{4(1-a)}. \tag{5.8}$$

This follows from Elliott's formula (5.5) and contains the classical relation of Legendre (5.3) as a special case.

Finally, we record the following formula of Kummer [Kum, p. 63, Form. 30]:

$$\begin{aligned} &F(a,b;a+b-c+1;1-x)F(a+1,b+1;c+1;x) \\ &\quad + \frac{c}{a+b-c+1}F(a,b;c;x)F(a+1,b+1;a+b-c+2;1-x) \\ &\quad = Dx^{-c}(1-x)^{c-a-b-1}, \quad D = \frac{\Gamma(a+b-c+1)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)}. \end{aligned}$$

This formula, like Elliott's identity, may be rewritten in many different ways if we use the contiguous relations of Gauss. Note also the special case $c = a+b-c+1$.

6. Mean values

The *arithmetic-geometric mean* of positive numbers $a, b > 0$ is the limit

$$AGM(a,b) = \lim a_n = \lim b_n,$$

where $a_0 = a$, $b_0 = b$, and for $n = 0, 1, 2, 3, \ldots$,

$$a_{n+1} = A(a_n, b_n) \equiv (a_n + b_n)/2, \quad b_{n+1} = G(a_n, b_n) \equiv \sqrt{a_n b_n},$$

are the arithmetic and geometric means of a_n and b_n, resp. For a mean value M, we also consider the *t-modification* defined as

$$M_t(a,b) = M(a^t, b^t)^{1/t}.$$

For example, the *power mean* of $a, b > 0$ is

$$A_t(a,b) = \left(\frac{a^t + b^t}{2}\right)^{1/t},$$

and the *logarithmic mean* is

$$L(a,b) = \frac{a-b}{\log(b/a)}.$$

The power mean is the t-modification of the arithmetic mean $A_1(a,b)$.

The connection between mean values and elliptic integrals is provided by Gauss's amazing result

$$AGM(1,r') = \frac{\pi}{2\mathcal{K}(r)}.$$

This formula motivates the question of finding minorant/majorant functions for $\mathcal{K}(r)$ in terms of mean values. For a fixed $x > 0$ the function $t \mapsto L_t(1,x), t > 0$, increases with t by [VV, Theorem 1.2 (1)]. The two-sided inequality

$$L_{3/2}(1,x) > AG(1,x) > L(1,x)$$

holds; the second inequality was pointed out in [CV], and the first one, due to J. and P. Borwein [BB2], proves a sharp estimate settling a question raised in connection with [VV]. Combined with the identity above, this inequality yields a very precise inequality for $\mathcal{K}(r)$.

Several inequalities between mean values have been proved recently. See, for instance, [AlQ2], [QS], [S1], [S2], [S3], [T], [C], and [Br].

Finally, we remark that the arithmetic-geometric mean, together with Legendre's Relation, played a central role in a rapidly converging algorithm for the number π in [Sa]. See also [BB1, H, Le, Lu].

7. Approximation of elliptic integrals

Efficient algorithms for the numerical evaluation of $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are based on the arithmetic-geometric mean iteration of Gauss. This fact led to some close majorant/minorant functions for $\mathcal{K}(r)$ in terms of mean values in [VV].

Next, let a and b be the semiaxes of an ellipse with $a > b$ and eccentricity $e = \sqrt{a^2 - b^2}/a$, and let $L(a,b)$ denote the arc length of the ellipse. Without loss of generality we take $a = 1$. In 1742, Maclaurin (cf. [AB]) determined that

$$L(1,b) = 4\mathcal{E}(e) = 2\pi \cdot {}_2F_1(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2).$$

In 1883, Muir (cf. [AB]) proposed that $L(1,b)$ could be approximated by the expression $2\pi[(1 + b^{3/2})/2]^{2/3}$. Since this expression has a close resemblance to the power mean values studied in [VV], it is natural to study the sharpness of this approximation. Close numerical examination of the error in this approximation led Vuorinen [V2] to conjecture that Muir's approximation is a lower bound for the arc length. Letting $r = \sqrt{1-b^2}$, Vuorinen asked whether

$$\frac{2}{\pi}\mathcal{E}(r) = {}_2F_1\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; r^2\right) \geqslant \left(\frac{1 + (r')^{3/2}}{2}\right)^{2/3} \tag{7.1}$$

for all $r \in [0, 1]$.

In [BPR1] Barnard and his coauthors proved that inequality (7.1) is true. In fact, they expanded both functions into Maclaurin series and proved that the differences of the corresponding coefficients of the two series all have the same sign.

Later, the same authors [BPR2] discovered an upper bound for $\mathcal{E}$ that complements the lower bound in (7.1):

$$\frac{2}{\pi}\mathcal{E}(r) = {}_2F_1\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; r^2\right) \leqslant \left(\frac{1 + (r')^2}{2}\right)^{1/2}, \quad 0 \leqslant r \leqslant 1. \tag{7.2}$$

See also [BPS].

In [BPR2] the authors have considered 13 historical approximations (by Kepler, Euler, Peano, Muir, Ramanujan, and others) for the arc length of an ellipse and determined a linear ordering among them. Their main tool was the following Lemma 7.3 on generalized hypergeometric functions. These functions are defined by the formula

$${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; z) \equiv 1 + \sum_{n=1}^{\infty} \frac{\Pi_{i=1}^{p}(a_i, n)}{\Pi_{j=1}^{q}(b_j, n)} \cdot \frac{z^n}{n!},$$

where p and q are positive integers and in which no denominator parameter b_j is permitted to be zero or a negative integer. When $p = 2$ and $q = 1$, this reduces to the usual Gaussian hypergeometric function $F(a, b; c; z)$.

7.3. **Lemma.** *Suppose $a, b > 0$. Then for any ϵ satisfying $\frac{ab}{1+a+b} < \epsilon < 1$,*

$${}_3F_2(-n, a, b; 1 + a + b, 1 + \epsilon - n; 1) > 0$$

for all integers $n \geqslant 1$.

7.4. **Some inequalities for $\mathcal{K}(r)$.** At the end of the preceding section we pointed out that upper and lower bounds can be found for $\mathcal{K}(r)$ in terms of mean values. Another source for the approximation of $\mathcal{K}(r)$ is based on the asymptotic behavior at the singularity $r = 1$, where $\mathcal{K}(r)$ has logarithmic growth. Some of the approximations motivated by this aspect will be discussed next.

Anderson, Vamanamurthy, and Vuorinen [AVV2] approximated $\mathcal{K}(r)$ by the inverse hyperbolic tangent function arth, obtaining the inequalities

$$\frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\,\frac{\operatorname{arth} r}{r}, \tag{7.5}$$

for $0 < r < 1$. Further results were proved by Laforgia and Sismondi [LS]. Kühnau [Ku] and Qiu [Q] proved that, for $0 < r < 1$,

$$\frac{9}{8 + r^2} < \frac{\mathcal{K}(r)}{\log(4/r')}.$$

Qiu and Vamanamurthy [QVa] proved that

$$\frac{\mathcal{K}(r)}{\log(4/r')} < 1 + \frac{1}{4}(r')^2 \quad \text{for } 0 < r < 1.$$

Several inequalities for $\mathcal{K}(r)$ are given in [AVV5, Theorem 3.21]. Later Alzer [A3] showed that

$$1+\Big(\frac{\pi}{4\log 2}-1\Big)(r')^2<\frac{\mathcal{K}(r)}{\log(4/r')},$$

for $0<r<1$. He also showed that the constants $\frac{1}{4}$ and $\pi/(4\log 2)-1$ in the above inequalities are best possible.

For further refinements, see [QVu1, (2.24)] and [Be].

Alzer and Qiu [AlQ1] have written a related manuscript in which, besides proving many inequalities for complete elliptic integrals, they have refined (7.5) by proving that

$$\frac{\pi}{2}\Big(\frac{\operatorname{arth} r}{r}\Big)^{3/4}<\mathcal{K}(r)<\frac{\pi}{2}\,\frac{\operatorname{arth} r}{r}.$$

They also showed that $3/4$ and 1 are the best exponents for $(\operatorname{arth} r)/r$ on the left and right, respectively.

One of the interesting tools of these authors is the following lemma of Biernaki and Krzyż [BK] (for a detailed proof see [PV1]):

7.6. **Lemma.** *Let r_n and s_n, $n=1,2,\dots$ be real numbers, and let the power series $R(x)=\sum_{n=1}^{\infty}r_nx^n$ and $S(x)=\sum_{n=1}^{\infty}s_nx^n$ be convergent for $|x|<1$. If $s_n>0$ for $n=1,2,\dots$, and if r_n/s_n is strictly increasing (resp. decreasing) for $n=1,2,\dots$, then the function R/S is strictly increasing (resp. decreasing) on $(0,1)$.*

7.7. **Generalized elliptic integrals.** For the case of generalized elliptic integrals some inequalities are given in [AQVV]. B. C. Carlson has introduced some standard forms for elliptic integrals involving certain symmetric integrals. Approximations for these functions can be found in [CG].

8. Landen inequalities

It is well known (cf. [BF]) that the complete elliptic integral of the first kind satisfies the Landen identities

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathcal{K}(r),\quad \mathcal{K}\left(\frac{1-r}{1+r}\right)=\frac{1+r}{2}\mathcal{K}'(r).$$

Recall that $\mathcal{K}(r)=\frac{\pi}{2}F(\frac{1}{2},\,\frac{1}{2};1;r^2)$. It is thus natural to consider, as suggested in [AVV3], the problem of finding an analogue of these formulas for the zero-balanced hypergeometric function $F(a,b;c;r)$ for $a,b,c>0$ and $a+b=c$, at least when the parameters (a,b,c) are close to $(\frac{1}{2},\,\frac{1}{2},1)$. From (3.3) it is clear that $F(a,b;c;r^2)$ has a logarithmic singularity at $r=1$, if $a,b>0$, $c=a+b$ (cf. [AAR]). Some refinements of the growth estimates were given in [ABRVV] and [PV1].

Qiu and Vuorinen [QVu1] proved the following Landen-type inequalities: For $a,b\in(0,1)$, $c=a+b$,

$$\begin{aligned}F\bigg(a,b;c;\Big(\frac{2\sqrt{r}}{1+r}\Big)^2\bigg)&\leqslant(1+r)F(a,b;c;r^2)\\&\leqslant F\bigg(a,b;c;\Big(\frac{2\sqrt{r}}{1+r}\Big)^2\bigg)+\frac{1}{B}(R-\log 16)\end{aligned}$$

and

$$\frac{1+r}{2}F(a,b;c;1-r^2) \leqslant F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^2\right)$$
$$\leqslant \frac{1+r}{2}\left[F(a,b;c;1-r^2)+\frac{1}{B}(R-\log 16)\right],$$

with equality in each instance if and only if $a=b=\frac{1}{2}$. Here $B=B(a,b)$, the beta function, and $R=R(a,b)=-2\gamma-\Psi(a)-\Psi(b)$, where Ψ is as given in Section 2.

9. Hypergeometric series as an analytic function

For rational triples (a,b,c) the hypergeometric function often can be expressed in terms of elementary functions. Long lists with such triples containing hundreds of functions can be found in [PBM]. For example, the functions

$$f(z) \equiv zF(1,1;2;z) = -\log(1-z)$$

and

$$g(z) \equiv zF(1,1/2;3/2;z^2) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

have the property that they both map the unit disk into a strip domain. Observing that they both correspond to the case $c=a+b$ one may ask (see [PV1, PV2]) whether there exists $\delta>0$ such that $zF(a,b;a+b;z)$ and $zF(a,b;a+b;z^2)$ with $a,b\in(0,\delta)$ map into a strip domain.

Membership of hypergeometric functions in some special classes of univalent functions is studied in [PV1, PV2, BPV2].

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ANDERSON:
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA
email: anderson@math.msu.edu
FAX: +1-517-432-1562

VAMANAMURTHY:
Department of Mathematics
University of Auckland
Auckland, NEW ZEALAND
email: vamanamu@math.auckland.nz
FAX: +649-373-7457

VUORINEN:
Department of Mathematics
University of Helsinki
P.O. Box 4 (Yliopistonkatu 5)
FIN-00014, FINLAND
e-mail: vuorinen@csc.fi
FAX: +358-9-19123213